EQUIVALENCE OF GEOMETRIC AND COMBINATORIAL DEHN FUNCTIONS

José Burillo and Jennifer Taback

Abstract. We prove that if a finitely presented group acts properly discontinuously, cocompactly and by isometries on a simply connected Riemannian manifold, then the Dehn function of the group and the corresponding filling function of the manifold are equivalent, in a sense described below.

1. Dehn functions and their equivalence

Let X be a simply connected 2-complex , and let w be an edge circuit in $X^{(1)}$. If D is a van Kampen diagram for w (see [5]), then the area of D is defined as the number of 2-cells on D, and the area of w, denoted $a(w)$, is defined as the minimum of the areas of all van Kampen diagrams for w. The Dehn function of X is then defined to be

$$\delta_X(n) = \max a(w),$$

where the maximum is taken over all loops w of length $l(w) \leq n$.

Given two functions f and g from $\mathbb{N}$ to $\mathbb{N}$ (or, more generally, from $\mathbb{R}^+$ to $\mathbb{R}^+$), we say that $f \prec g$ if there exist positive constants A, B, C, D, E so that

$$f(n) \leq Ag(Bn + C) + Dn + E.$$

Two such functions are called equivalent (denoted $f \equiv g$) if $f \prec g$ and $g \prec f$. The Dehn function is invariant under quasi-isometries: when one considers the 1-skeleton of a complex as a metric space with the path metric, where every edge has length one, two complexes with quasi-isometric 1-skeleta have equivalent Dehn functions (see [1]).

Let G be a finitely presented group, and let $\mathcal{P}$ be a finite presentation for G. Let $K = K(\mathcal{P})$ be the 2-complex associated to $\mathcal{P}$, i.e. the 2-complex with a single vertex, an oriented edge for every generator of $\mathcal{P}$, and a 2-cell for every relator, attached to the edges according to the spelling of the relator. Then the Dehn function of $\mathcal{P}$ is, by definition, the Dehn function $\delta_{\tilde{K}}$ of the universal covering of K. Two finite presentations $\mathcal{P}$ and $\mathcal{Q}$ for the same group G yield 2-complexes $\widetilde{K}(\mathcal{P})$ and $\widetilde{K}(\mathcal{Q})$ with quasi-isometric 1-skeleta, and hence equivalent Dehn functions. Thus the Dehn function of the group G is defined to be the equivalence class of the Dehn function of any of its presentations. An extensive treatment of Dehn functions of finitely presented groups is given in [4].

A closely related definition can be formulated in the context of Riemannian manifolds, dating back to the isoperimetric problem for $\mathbb{R}^n$ in the calculus of variations. Given a Lipschitz loop γ in a simply connected Riemannian manifold M, we define

the area of γ to be the infimum of the areas of all Lipschitz discs bounded by γ. We then define the geometric Dehn function of M by

$$\delta_M(x) = \max_{l(\gamma)\leq x} \text{area}(\gamma)$$

where $l(\gamma) = length(\gamma)$.

It is natural to consider the question of whether the Dehn functions of a simply connected Riemannian manifold M and of a finitely presented group G acting properly discontinuously and cocompactly on M agree. The fact that they effectively agree has been implicitly assumed in the literature, though no proof has been given. A closely related statement is given in [2, Theorem 10.3.3], applying the Deformation Theorem of Geometric Measure Theory ([3, 4.2.9] and [7]) to this setting, and which provides the basis of the Pushing Lemma below. This paper is devoted to providing a complete and detailed proof that the combinatorial and geometric Dehn functions are equivalent. It is known to the authors that M. Bridson has lectured on an alternate, unpublished proof for the same result. The authors would like to thank Professor S. M. Gersten for his encouragement and his useful remarks, Kevin Whyte for helpful conversations and the referee for his precise comments.

We prove the following theorem.

Theorem 1.1. *Let M be a simply connected Riemannian manifold, and G a finitely presented group acting properly discontinuously, cocompactly and by isometries on M. Let τ be a G-invariant triangulation of M. Then the following three Dehn functions are equivalent:*

1. *the Dehn function δ_G of any finite presentation of G,*
2. *the Dehn function $\delta_{\tau^{(2)}}$ of the 2-skeleton of τ, and*
3. *the geometric Dehn function δ_M of M.*

The fact that δ_G and $\delta_{\tau^{(2)}}$ are equivalent is clear: since G acts cocompactly on τ, there is a quasi-isometry between $\tau^{(1)}$ and the 1-skeleton of $\widetilde{K}(\mathcal{P})$ for any presentation $\mathcal{P}$ of G, and the equivalence follows from the results in [1]. We will concentrate on the proof of the equivalence between $\delta_{\tau^{(2)}}$ and δ_M. The arguments will be mainly geometric, relating the lengths and areas of loops and discs in M with those included in the triangulation τ. The first step in this direction is the Pushing Lemma, which is a complete analog of the Deformation Theorem in Geometric Measure Theory and already stated and proved, in a slightly different way, in [2, Theorem 10.3.3], and whose proof we will follow closely.

2. Technical Lemmas

The Pushing Lemma, stated below, will allow us to relate arbitrary Lipschitz chains in M to chains in the corresponding skeleta of τ. The main technical problem to be overcome is that projection of a Lipschitz chain to τ from a badly chosen point can increase the volume of the chain arbitrarily. We overcome this by using techniques from measure theory that assure the existence of a center of projection far enough from the chain, thus providing control on the growth of the volume.

Lemma 2.1 (Pushing Lemma). *Let M, G and τ be as above. Then there exists a constant C, depending only on M and τ, with the following property: Let T be a Lipschitz k-chain in M, such that ∂T is included in $\tau^{(k-1)}$. Then there exists another Lipschitz k-chain R, with $\partial R = \partial T$, which is included in $\tau^{(k)}$, and a Lipschitz $(k+1)$-chain S, with $\partial S = T - R$, satisfying*

$$\mathrm{vol}_k(R) \leq C\mathrm{vol}_k(T) \qquad \textit{and} \qquad \mathrm{vol}_{k+1}(S) \leq C\mathrm{vol}_k(T).$$

In particular, if T is a loop, so is R, and S is a homotopy from T to R.

The Pushing Lemma differs from the statement in [2] because it applies to chains as well as cycles, since the boundary of the chain is not modified, as it is included in the $(k-1)$-skeleton. A statement for cycles is not sufficient, since this lemma will be applied to chains as well as loops, and the fact that $\partial T = \partial R$ is crucial in the proof of the main theorem.

We first prove a lemma which will later allow us to choose our center of projection to lie away from the Lipschitz chain T.

Lemma 2.2. *Let $f : S^k \to \sigma_{k+1}$ be Lipschitz with constant L, where σ_{k+1} is the standard Euclidean $(k+1)$-simplex. Then $f(S^k)$ has Lebesgue $(k+1)$-measure zero.*

Proof. Since S^k is compact, choose a finite open cover of S^k by k-dimensional balls B_i of radius $\frac{1}{n}$. We can cover S^k with $C_1 n^k$ such balls, for some constant C_1. The image of any ball B_i under the Lipschitz map f is contained in a $(k+1)$-dimensional ball $B'_i \subset \sigma_{k+1}$ with $(k+1)$-volume $\frac{C_2}{n^{k+1}}$ for some constant C_2. Then the total volume of the collection $\{B'_i\}$ is at most $\frac{C_1C_2}{n}$. So $f(S^k)$ is contained in an open set of σ_{k+1} whose total volume is $\frac{C_1C_2}{n}$ and thus $f(S^k)$ has Lebesgue measure 0.

Proof of Lemma 2.1. The proof will proceed by descending induction on the skeleta of τ. Assume that a Lipschitz k-chain T is included in $\tau^{(i)}$ but not in $\tau^{(i-1)}$, for $i > k$. We want to proceed simplex by simplex, choosing an appropriate point not in T in each simplex and projecting the chain T radially from this point to the boundary of the simplex. We will prove the following claim.

Claim: There exists a constant C with the property that for every simplex, there is a point p not in T so that radial projection of T from p to the boundary of the simplex does not increase the volume of the chain by more than a multiplicative factor C.

Observe that since T is compact, it only intersects finitely many simplices of τ, and in each simplex is only modified by a radial projection from a point not in T. These radial projections only increase the Lipschitz constant of T, but the chain R obtained after the projections will still be Lipschitz.

To simplify the computations, we will work through the proof in the unit Euclidean simplex of edge length one. Since G acts cocompactly on M, we can construct a sufficiently fine finite triangulation of the quotient and lift it to M. If the simplices are small enough we can map them to $\mathbb{R}^n$ via the exponential map. Since the exponential map is Lipschitz, the changes in the metric are bounded by only a multiplicative constant. We then have a finite number of simplices in $\mathbb{R}^n$, so the distortion is again bounded. Thus working with the unit simplex only affects the value of the constant C.

Let σ be the unit Euclidean i-simplex, O the barycenter of σ, and r a positive number so that the ball of center O and radius $3r$ is included in the interior of σ. Let B be the ball of center O and radius r, with u an element of B, and B_u the ball of center u and radius $2r$. Clearly $B \subset B_u$, for all u. Let π_u be the radial projection with center u of $B_u \setminus \{u\}$ onto ∂B_u. Let $Q = T \cap \sigma$. We want to see that there exists a constant v_0 independent of T and σ, and a point $u \in B \setminus Q$, dependent on T, with

$$\mathrm{vol}_k(\pi_u Q) \le v_0 \, \mathrm{vol}_k(Q).$$

From Lemma 2.2, we see that the set $B \setminus Q$ has the same measure as B, allowing us to choose $u \in B \setminus Q$.

For every positive real number v define

$$A_v = \{u \in B \setminus Q \mid \mathrm{vol}_k(\pi_u Q) > v \, \mathrm{vol}_k(Q)\}$$

and let $\alpha(v) = m_i(A_v)$, where m_i is the i-dimensional Lebesgue measure. We want to prove that

$$\lim_{v \to \infty} \alpha(v) = 0.$$

Then we will choose v_0 with $\alpha(v_0) < m_i(B)$, so the measure of A_{v_0} will be less than the measure of B. Thus there will exist a point $u \in (B \setminus Q) \setminus A_{v_0}$, which will be the center of projection. Since $u \notin A_{v_0}$, this projection will increase the area at most by a multiplicative factor v_0.

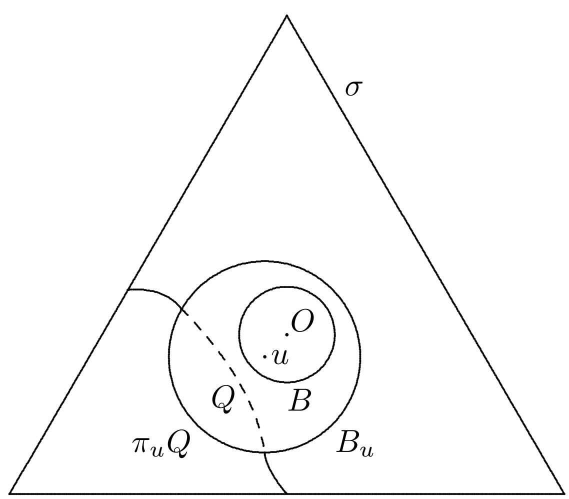

Figure 1: *Projecting Q to the boundary of B_u.*

We have

$$\begin{aligned}\mathrm{vol}_k(\pi_u Q) &\le \mathrm{vol}_k(\pi_u(Q \cap B_u)) + \mathrm{vol}_k(Q) \\ &\le \int_{Q \cap B_u} \left(\frac{2r}{||x-u||} \right)^k dx + \mathrm{vol}_k(Q),\end{aligned}$$

where the first term accounts for the volume obtained after projecting, and the second term takes care of the possibility of Q and B_u being disjoint. Assume now

that $\mathrm{vol}_k(Q)$ is nonzero (if $\mathrm{vol}_k(Q) = 0$ then $\mathrm{vol}_k(\pi_u Q) = 0$). Then we have:

$$\begin{aligned}
\alpha(v)\, v\, \mathrm{vol}_k(Q) &= v\, \mathrm{vol}_k(Q) \int_{A_v} du = \int_{A_v} v\, \mathrm{vol}_k(Q)\, du \\
&\leq \int_{A_v} \mathrm{vol}_k(\pi_u Q)\, du \leq \int_B \mathrm{vol}_k(\pi_u Q)\, du \\
&\leq \int_B \left(\int_{Q\cap B_u} \left(\frac{2r}{||x-u||} \right)^k dx + \mathrm{vol}_k(Q) \right) du \\
&= (2r)^k \int_{Q\cap B_u} \int_B ||u-x||^{-k}\, du\, dx + \mathrm{vol}_i(B)\mathrm{vol}_k(Q).
\end{aligned}$$

Notice that the function $\left(\frac{2r}{||x-u||}\right)^k$ is bounded above and below, since $u \notin Q \cap B_u$, and is integrated over compact regions. This allows us to change the order of integration. Now make a change of variables, letting $w = u - x$, and increase the domain of integration to $B(0, 3r)$. We continue with the upper bound for $\alpha(v)\, v\, \mathrm{vol}_k(Q)$:

$$\begin{aligned}
\alpha(v)\, v\, \mathrm{vol}_k(Q) &\leq (2r)^k \int_{Q\cap B_u} \int_B ||u-x||^{-k}\, du\, dx + \mathrm{vol}_i(B)\mathrm{vol}_k(Q) \\
&\leq (2r)^k \int_{Q\cap B_u} dx \int_{B(O,3r)} ||w||^{-k}\, dw + \mathrm{vol}_i(B)\mathrm{vol}_k(Q) \\
&\leq K \mathrm{vol}_k(Q),
\end{aligned}$$

where

$$K = (2r)^k \int_{B(O,3r)} ||w||^{-k}\, dw + \mathrm{vol}_i(B).$$

Observe that K is finite and independent of T and σ. We conclude that $\alpha(v)v \leq K$. Knowing K, we can find v_0 such that $K/v_0 < m_i(B)$, where v_0 is a constant independent of T and σ. We have now found A_{v_0} with strictly less measure than B, and can pick a point in $(B \setminus Q) \setminus A_{v_0}$ from which to project so that the volume increases at most by a multiplicative factor v_0.

The result of the above argument is the construction of another chain $\pi_u Q$ which is far enough from O. We can now project radially from O to $\partial\sigma$, and the change of volume is bounded since $\pi_u Q$ is at least at a distance r from O. The combination of this change of volume with v_0 gives the constant needed in this precise skeleton. Combining the constants from all of these steps, we obtain the desired constant C. Observe that these projections leave $\tau^{(i-1)}$ unchanged, so clearly ∂T is preserved.

The $(k+1)$-chain S is obtained by joining every $x \in Q$ to $\pi_u x$ by a segment. The volume of the piece of S contained in σ is then bounded, as before, by

$$(2r)^{k+1} \int_{Q\cap B_u} \frac{dx}{||x-u||^k},$$

where the extra factor $2r$ is obtained from the direction of the projection, since each segment has length bounded by $2r$. An argument similar to the previous one shows that projecting from most points in B gives the right bound for the volume.
□

The third lemma states that for a Lipschitz map, almost every point in the target space has a finite number of preimages. It is a direct consequence of the area formula for Lipschitz maps, and it will be used for both loops and discs in the proof of Theorem 1.1.

Lemma 2.3. *Let T be a Lipschitz k-chain in M, where $k \leq \dim M$. Then the set of points in M with infinite preimages under T has Hausdorff k-measure zero.*

Proof. Let σ_k be the standard closed k-simplex, and let

$$E : \sigma_k \longrightarrow M$$

be one of the simplices in T. Since E is a Lipschitz map, by Rademacher's Theorem ([3, 3.1.6]) it is differentiable almost everywhere (with respect to the Lebesgue k-measure), so the Jacobian $J_k E(x)$ is well defined for almost all $x \in \sigma_k$. For $y \in M$, let $N(E, y)$ be the number of elements of $E^{-1}(y)$, possibly infinite, and denote by m_k and h_k the Lebesgue and Hausdorff k-measures, respectively. Then the area formula for Lipschitz maps ([3, 3.2.3]) states that

$$\int_{\sigma_k} |J_k E(x)| \, dm_k(x) = \int_M N(E, y) \, dh_k(y).$$

Since E is Lipschitz, we know that $|J_k E(x)|$ is bounded, and since σ_k has finite measure, the integral on the left hand side is finite. So the set where $N(E, y)$ is infinite cannot have positive Hausdorff k-measure, because then the right hand side of the equation would be infinite. □

3. Proof of the Main Theorem

We begin by proving the one of the two inequalities necessary for the equivalence of δ_M and $\delta_{\tau^{(2)}}$, namely

$$\delta_M \prec \delta_{\tau^{(2)}}. \tag{3.1}$$

Let γ be a Lipschitz loop in M, with length at most n. Using the Pushing Lemma, we can construct a new loop η, of length at most Cn, which is included in the 1-skeleton, and the homotopy between γ and η has area at most Cn.

The loop η is not necessarily combinatorial, but it is a rectifiable loop in a non-positively curved space, namely the metric graph $\tau^{(1)}$. So there is a unique (up to reparametrization) closed geodesic ζ in the free homotopy class of η. The straight homotopy (in $\tau^{(1)}$) from η to ζ is a map from an annulus to $\tau^{(1)}$. The length of ζ decreases monotonically and its area can be made arbitrarily small.

The combinatorial loop ζ can be filled combinatorially by at most $\delta_{\tau^{(2)}}(Cn)$ 2-simplices in τ. Thus

$$\delta_M(n) \leq A\delta_{\tau^{(2)}}(Cn) + 2Cn,$$

where A is the area of the largest 2-cell in τ, and it follows that $\delta_M \prec \delta_{\tau^{(2)}}$.

To prove the reverse inequality

$$\delta \prec \delta$$

to (3.1), we start with a combinatorial loop γ in the 1-skeleton of τ, with length at most n. Let

$$f : D^2 \longrightarrow M$$

be a Lipschitz disc in M with boundary γ, and with area a. We want to construct a van Kampen diagram for γ and bound its area in terms of a. The first step is to use the Pushing Lemma to find a new disc (also denoted f) which is included in $\tau^{(2)}$, and whose area is at most Ca.

Let σ be an open 2-simplex of τ contained in $f(D^2)$. By Lemma 2.3, we can choose a point $p \in \sigma$, such that $f^{-1}(p)$ is finite. Let X be a component of $f^{-1}(\sigma)$. If $X \cap f^{-1}(p) = \varnothing$, then $f\big|_X$ can be modified by composing with a radial projection from p. After this change, a component X of $f^{-1}(\sigma)$ satisfies $X \cap f^{-1}(p) \neq \varnothing$, and there are only finitely many of these components. Moreover, if $f\big|_X$ is not surjective, we can again modify $f\big|_X$ by a radial projection from a point not in $f(X)$, to push its image to $\partial\sigma$. After these changes to f, there is a component X of $f^{-1}(\sigma)$ so that $f\big|_X$ is surjective, and $X \cap f^{-1}(p) \neq \varnothing$. If X is one such component, the original f has not been modified in X by any radial projection, and the map

$$f\big|_X : X \longrightarrow \sigma$$

is still Lipschitz, since it is the restriction of the original map f.

We will obtain a lower bound on the area of $f\big|_X$ using the degree of $f\big|_X$. Since $f\big|_X$ is differentiable almost everywhere, we can define the degree of $f\big|_X$ at a point $y \in f(X)$ by

$$\deg f\big|_X(y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} J_2 f(x).$$

Moreover, since X is an open connected component of $f^{-1}(\sigma)$, we have that $f(X) \subset \sigma$ and $f(\partial X) \subset \partial\sigma$, so $f(X)$ and $f(\partial X)$ are disjoint. Then, by [3, 4.1.26], the degree of $f\big|_X$ is almost constant in $f(X)$, and we can define the degree of $f\big|_X$ as the value d_X it achieves at almost every $y \in f(X)$. The lower bound on the area of $f\big|_X$ is given by the area formula for Lipschitz maps: if u is an integrable function with respect to m_2, we have (see [3, 3.2.3]):

$$\int_X u(x) |J_2 f(x)|\, dm_2 = \int_\sigma \sum_{x \in f^{-1}(y) \cap X} u(x)\, dh_2,$$

and taking $u(x) = \operatorname{sign} Jf(x)$ we obtain:

$$\operatorname{area} f\big|_X = \int_X |J_2 f(x)|\, dm_2 \geq \left| \int_X J_2 f(x)\, dm_2 \right| =$$
$$\left| \int_X \operatorname{sign} J_2 f(x)\, |J_2 f(x)|\, dm_2 \right| = \left| \int_\sigma \deg f\big|_X\, dh_2 \right| = \frac{\sqrt{3}}{4} |d_X|.$$

Our goal is to find a simplicial map

$$g : D^2 \longrightarrow \tau^{(2)}$$

(with some simplicial structure in D^2) such that only $|d_X|$ simplices are mapped by the identity to σ under $g\big|_X$, and the rest of X is mapped to $\partial\sigma$. Then we will have that the combinatorial area of g is bounded as follows,

$$\sum_X |d_X| \le \sum_X \frac{4}{\sqrt{3}}\text{area}\left(f\big|_X\right) \le \frac{4}{\sqrt{3}}Ca$$

giving us the required bound. Note that the map g is not combinatorial, but only simplicial, and at the end of the proof a short argument will be required to ensure the existence of a combinatorial map whose area admits the same upper bound.

The first step in finding the map g is to smooth the map $f\big|_X$, in order to apply differentiable techniques to it. Let O be the barycenter of σ, and choose $0 < \epsilon < r$ such that:

$$\varnothing \ne B(O, r-\epsilon) \subset B(O,r) \subset B(O,2r) \subset B(O,2r+\epsilon) \subset \sigma,$$

and let $U_1 = f^{-1}(B(O,r))$ and $U_2 = f^{-1}(B(O,2r))$. We have that $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset X$. Choose $\delta > 0$ so that $B(x,\delta) \subset X$ for all $x \in U_2$, and so that if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$, for all $x, y \in X$. Let φ be a C^∞ bump function in $\mathbb{R}^2$ with support in $B(0,\delta)$, and with integral 1. Then, for $x \in U_2$, we can construct the convolution

$$f * \varphi(x) = \int_{B(x,\delta)} f(x-z)\varphi(z)\,dz,$$

which is C^∞ in U_2, and satisfies $|f(x) - f*\varphi(x)| < \epsilon$ for all $x \in U_2$. Also, if $f\big|_X$ was Lipschitz with constant L, then $f*\varphi$ is also Lipschitz with the same constant: if $x, y \in U_2$,

$$|f*\varphi(x) - f*\varphi(y)| \le |f(x-z) - f(y-z)| \int_{B(0,\delta)} \varphi(z)\,dz \le L|x-y|.$$

Now choose a Lipschitz function α on X with values in $[0,1]$ which is equal to 1 in U_1 and equal to 0 outside U_2, and define

$$\tilde{f} = \alpha(f*\varphi) + (1-\alpha)f\big|_X.$$

Note that $\tilde{f}$ is defined only on X. Then $\tilde{f}$ satisfies the following properties:

(1) $|f(x) - \tilde{f}(x)| < \epsilon$ for all $x \in X$,
(2) $\tilde{f}$ is smooth in U_1,
(3) $\tilde{f} = f$ in $X \setminus U_2$,
(4) $\tilde{f}$ is Lipschitz, and
(5) $\deg \tilde{f} = \deg f\big|_X$.

The first three properties are clear from the construction of $\tilde{f}$, and property (4) holds because $f\big|_X$ and $f*\varphi$ and α are all Lipschitz. To see that the degree is unchanged, since the degree is almost constant, and $f\big|_X$ and $\tilde{f}$ agree outside U_2, we only need to find a point in $\sigma \setminus B(O, 2r+\epsilon)$ for which the degree is d_X for both $f\big|_X$ and $\tilde{f}$.

We can now use Sard's Theorem ([6]) to claim the existence of a regular value

let $p_1, \ldots, p_m$ be its preimages. Let V be an open disc with center q such that $\tilde{f}^{-1}(V) = V_1 \cup \ldots \cup V_m$, where the V_i are discs around p_i, pairwise disjoint, and such that $\tilde{f}\big|_{V_i}$ is a diffeomorphism. In general, we will have that $m > |d_X|$, and must cancel discs with opposite orientations. Assume V_{m-1} and V_m are mapped to V with opposite orientations. Choose $a \in \partial V_{m-1}$ and $a' \in \partial V_m$ with $\tilde{f}(a) = \tilde{f}(a')$, and join a and a' with a simple path λ such that $\tilde{f}(\lambda)$ is nullhomotopic in $\sigma \setminus V$. This can be done because the map

$$\tilde{f} : X \setminus \bigcup_{i=1}^{m} V_i \longrightarrow \sigma \setminus V$$

induces a surjective homomorphism of fundamental groups. After contracting $\tilde{f}(\lambda)$, we can assume $\tilde{f}(\lambda)$ is the constant path $\tilde{f}(a)$. Remove the discs V_{m-1} and V_m and perform surgery along λ. The new boundary thus created is mapped to ∂V under $\tilde{f}$ by a map from S^1 to itself of degree zero. Extend this map to a map from D^2 to S^1 and attach it to $\tilde{f}$ along this boundary. For the new map (which we will continue calling $\tilde{f}$), the preimage of q consists only of the points $p_1, \ldots, p_{m-2}$. Repeating this process we will obtain a map where now only the discs $V_1, \ldots, V_{|d_X|}$ are mapped to V, all with the same orientation.

Choose (temporarily) a sufficiently fine subdivision of τ so that there is a 2-simplex W in V, and let $\rho_i = \tilde{f}^{-1}(W)$. Modify the map in X by composing with the expansion of W into all of σ.

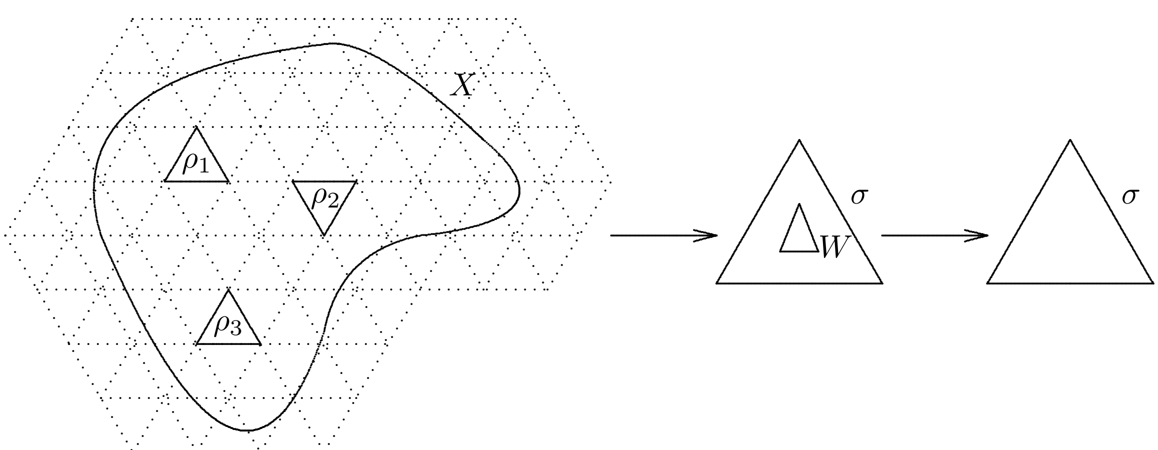

Figure 2: *Making the map f simplicial*

After this process is done for all σ, we obtain a map from D^2 to $\tau^{(2)}$, where all the ρ_i are sent homeomorphically to 2-simplices of τ, and the rest of D^2 is sent to the 1-skeleton of τ. To finish the construction of g, find a simplicial structure on D^2 compatible with the simplicial structure on the original loop γ and which includes all the ρ_i obtained for all σ as 2-simplices. Now approximate the map $\tilde{f}$ simplicially within $\tau^{(1)}$ relative to all the ρ_i and to γ. The result is simplicial, and the number of simplices sent by g homeomorphically to 2-simplices in τ is

$$\sum |d_X| \leq \frac{4}{\sqrt{3}} Ca.$$

This map is not a van Kampen diagram yet, since it is only simplicial. To finalize the proof of the inequality

$$\delta_{\tau^{(2)}} \prec \delta_M,$$

we will find a van Kampen diagram which satisfies the same upper bound as the map g. Consider simplicial maps from a contractible planar 2-complex Y into $\tau^{(2)}$, with boundary γ, whose area satisfies the same bound as g. (The map g shows the existence of such maps.) Among all these maps, choose one with the minimum number of 2-cells in Y. This map is necessarily combinatorial, since if some 2-cell of Y is collapsed to the 1-skeleton of $\tau^{(2)}$, we could collapse it in Y and find a map with fewer 2-cells. This map is the required van Kampen diagram for the loop γ, and the second inequality is proved.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain

E-mail address: burillo@mat.uab.es

Dept. of Mathematics and Statistics, University at Albany, Albany, NY 12222

E-mail address: jtaback@math.albany.edu